In (A) and (C) remove the brackets so as to unite the symbols of operation and the symbols of quantity and we have:

$$u_n=u_1+(n-1)\triangle u_1+\frac{(n-1)(n-2)}{2!}\triangle^2 u_1+\frac{(n-1)(n-2)(n-3)}{3!}\triangle^3 u_1+\ldots(D);$$

and

$$S_n=nu_1+\frac{n(n-1)}{2}\triangle u_1+\frac{n(n-1)(n-2)}{3!}\triangle^2 u_1$$
$$+\frac{n(n-1)(n-2)(n-3)}{4!}\triangle^3 u_1+\ldots(E).$$

In (D) and (E) substitute the values, $u_1=1$, $\triangle u_1=2$, $\triangle^2 u_1=2$, and $\triangle^3 u_1=4$, from the problem and its differences, and we have, after reduction:

$$u_n=\tfrac{1}{3}[2n^3-9n^2+19n-9]\ldots(F);\ \text{and}$$
$$S_n=\tfrac{1}{6}[n^4-4n^3+11n^2-2n]\ldots(G).$$

Equations (F) and (G) are true for all values of n for the special series under consideration. When $n=4$, $u_n=u_4=17$, and $S_n=S_4=28$, as may be seen by inspecting the series in the problem.

But equations (D) and (E) are perfectly general when the series follows any regular law of progression; as we have to know, only, the value of the leading term, and the leading differences up to the difference that vanishes, to find the value of any term in a series and the sum of that series.

II. **Solution by L. E. NEWCOMB, Los Gatos, Cal., and G. W. GREENWOOD, M. A., Dunbar, Pa.**

Let $S\equiv u_1+u_2x+u_3x^2+\ldots$ where $u_1=1$, $u_2=3$, $u_3=7$, and, in general, $u_n=2u_{n-1}+u_{n-2}$, x, of course, being less than unity, numerically.

$\therefore (1-2x-x^2)S=u_1+(u_2-2u_1)x$; *i. e.*,

$$S=\frac{1+x}{1-2x-x^2}=\frac{A}{1-\alpha x}+\frac{B}{1-\beta x}$$

where $\alpha=1+\sqrt{2}$, $\beta=1-\sqrt{2}$, $A=\frac{1}{2}(1+\sqrt{2})$, $B=\frac{1}{2}(1-\sqrt{2})$.

$\therefore u_n=A\alpha^{n-1}+B\beta^{n-1}=\frac{1}{2}[(1+\sqrt{2})^n+(1-\sqrt{2})^n]$.

Let $S_n=u_1+u_2+\ldots+u_n$; $S_n(1-2-1)=u_1+u_2-2u_1-3u_n-u_{n-1}$; *i. e.*, $S_n=\frac{1}{2}[3u_n+u_{n-1}-2]$.

Solved in a similar manner by J. Scheffer.

CALCULUS.

219. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate (a) $\int_0^{\frac{1}{2}\pi}\frac{\sin mx\sin nx}{\sin x}dx$; (b) $\int_0^{\frac{1}{2}\pi}\frac{\cos mx\sin nx}{\sin x}dx$, where n is a positive integer. Also, modify the result for the case of m an integer.

Solution by S. A. COREY, Hiteman, Iowa.

When n is a positive integer, sin nx may be developed into a sine power series divisible by sin x. Substituting this development in (a) or (b) each term may be readily integrated. This method is, of course, also applicable when m is an integer in (a), but when (b) m is an integer and n not an integer this method fails. In the latter case, as well as in the other cases, an approximate value of (a) and (b) may be deduced by the use of formula (1), page 12, AMERICAN MATHEMATICAL MONTHLY, January, 1906. By using no term higher than the third (the term involving B_2), and by obtaining $f'(x)$ and $f^{\mathrm{IV}}(x)$ by differentiating the right members of the following identities:

$$\text{(c)} \int\frac{\cos mx \sin nx}{\sin x}dx=\int\frac{\sin(m+n)x}{2\sin x}dx-\int\frac{\sin(m-n)x}{2\sin x}dx,$$

$$\text{(d)} \int\frac{\sin mx \sin nx}{\sin x}dx=\int\frac{\cos(m-n)x}{2\sin x}dx-\int\frac{\cos(m+n)x}{2\sin x}dx,$$

the following developments are obtained:

$$\text{(a)}=\frac{\pi}{2.2.r}\left\{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}+2\left[\frac{\sin(m\pi/2r)\ \sin(n\pi/2r)}{\sin(\pi/2r)}\right.\right.$$

$$\left.\left.+\frac{\sin(2m\pi/2r)\sin(2n\pi/2r)}{\sin(2\pi/2r)}+\ldots+\sin\frac{(r-1)m\pi}{2r}\sin\frac{(r-1)\,n\pi}{2r}\right]\right\}$$

$$-\frac{\pi^2}{6.2.2!(2r)^2}\left(s\sin\frac{s\pi}{2}-t\sin\frac{t\pi}{2}-2mn\right)$$

$$+\frac{\pi^4}{30.(2r)^4.2.4!}\left[(t^3-3t)\sin\frac{t\pi}{2}-(s^3-3s)\sin\frac{s\pi}{2}+2mn(m^2+n^2-1)\right]\ldots(1).$$

$$\text{(b)}=\frac{\pi}{2.2.r}\left\{\cos\frac{m\pi}{2}\sin\frac{n\pi}{2}+n+2\left[\frac{\cos(m\pi/2r)\ \sin(n\pi/2r)}{\sin(\pi/2r)}\right.\right.$$

$$\left.\left.+\frac{\cos(2m\pi/2r)\sin(2n\pi/2r)}{\sin(2\pi/2r)}+\ldots+\frac{\cos[(r-1)m\pi/2r]\ \sin[(r-1)n\pi/2r]}{\sin[(r-1)\pi/2r]}\right]\right\}$$

$$-\frac{\pi^2}{6.(2r)^2.2.2!}\left[s\cos\frac{s\pi}{2}-t\cos\frac{t\pi}{2}\right]$$

$$+\frac{\pi^4}{30.(2r)^4.2.4!}\left[(3s-s^3)\cos\frac{s\pi}{2}-(3t-t^3)\cos\right]\ldots(2),$$

where $s=(m+n)$, $t=(m-n)$. To insure rapid convergence, let $r>(m+n)$. If in (b), $(m+n)=(2p+1)$, m, n, and p integers, it is readily seen that the value of (b) is zero.

In order to test the accuracy of the work of computation as well as to test the convergence of the series, it is sometimes advisable to find the value of the definite integral with $r=2a$ after its value has been found with $r=a$. The work involved in this test is usually not great as the work that has been done when $r=a$ is made use of when $r=2a$.

To show the rapid convergence of (1) and (2) the two following simple examples will suffice:

$$\int_0^{\frac{1}{2}\pi}\frac{\cos(3x/2)\sin x}{\sin x}dx=\int_0^{\frac{1}{2}\pi}\cos\frac{3x}{2}dx=\tfrac{2}{3}\sqrt{\tfrac{1}{2}}.$$ Here $m=\frac{3}{2}$, $n=1$. Taking

$r=3$, we have $\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\cos(3x/2)\sin x}{\sin x}dx=\frac{\pi}{12}(1+\sqrt{\tfrac{1}{2}})+\frac{3\pi^2\sqrt{\frac{1}{2}}}{6^3.4}+\frac{7\pi^4\sqrt{\frac{1}{2}}}{30.6^4.2.4!}=.47141.$

Similarly, $\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\sin(3x/2)\sin x}{\sin x}=\frac{\pi}{12}(2+3\sqrt{\tfrac{1}{2}})+\frac{3\pi^2}{6^3.4}(1+\sqrt{\tfrac{1}{2}})+\frac{\pi^4}{30.6^4.2.4!}$

$\times(\frac{27}{4})(1+\sqrt{\frac{1}{2}})=1.13807$, both results being correct to five decimal places.

Also solved by G. B. M. Zerr.

DIOPHANTINE ANALYSIS.

137. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Prove that all multiply perfect numbers of multiplicity n having only n distinct primes are comprised in $n=2, 3, 4$.

Solution by JACOB WESTLUND, Ph. D., Purdue University, Lafayette, Ind.

If $p_1, p_2, \dots, p_n$ are the distinct prime factors of a number of multiplicity n, we must have $n<\prod_i^{1,n}\frac{p_i}{p_i-1}$, and hence $n<\frac{2}{1}.\frac{3}{2}.\frac{5}{4}.\frac{7}{6}\dots\frac{2n-1}{2n-2}$. But this is impossible when $n>4$, as seen by induction. For we have

$$(n+1)1.2.4.6\dots2n=n.1.2.4.6\dots2n+1.2.4.6\dots2n.$$

Now if $n.1.2.4.6\dots(2n-2)>2.3.5.7\dots(2n-1)$, it follows that

$$(n+1)1.2.4.6\dots2n>2.3.5.7\dots(2n-1)2n+1.2.4.6\dots2n, \text{ or}$$
$$(n+1)1.2.4.6\dots2n>2.3.5.7\dots(2n+1)-2.3.5.7\dots(2n-1)+1.2.4.6\dots2n.$$

Hence $(n+1)1.2.4.6\dots2n>2.3.5.7\dots(2n+1)$. For $n=5$ we have

$$5>\tfrac{2}{1}.\tfrac{3}{2}.\tfrac{5}{4}.\tfrac{7}{6}.\tfrac{9}{8}=\tfrac{315}{64}.$$

Hence for all values of $n>4$ we have $n>\prod_i^{1,n}\frac{p_i}{p_i-1}$, which proves the theorem.